

# COHEN-MACAULAY MONOMIAL IDEALS OF CODIMENSION 2

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**ABSTRACT.** We give a structure theorem for Cohen-Macaulay monomial ideals of codimension 2, and describe all possible relation matrices of such ideals. In case that the ideal has a linear resolution, the relation matrices can be identified with the spanning trees of a connected chordal graph with the property that each distinct pair of maximal cliques of the graph has at most one vertex in common.

*Key words* : Monomial Ideals, Taylor Complexes, Linear Resolutions, Chordal Graphs.

## INTRODUCTION

The purpose of the paper is to work out in detail a remark on the structure of Cohen-Macaulay monomial ideals of codimension 2 which was made in the paper [1]. There it was observed that the ‘generic’ ideals of this type, generated by  $n$  elements, are in bijective correspondence to the trees with  $n$  vertices. In Proposition 1.2 we give an explicit description of the generators of a generic Cohen-Macaulay monomial ideal of codimension 2 in terms of the associated tree and describe the minimal prime ideals of such ideals in Proposition 1.4. As a consequence of these two results we obtain as the main result of Section 1 a full description of all Cohen-Macaulay monomial ideals of codimension 2, see Theorem 1.5.

In Section 2 we study the possible relation trees of a Cohen-Macaulay monomial ideals of codimension 2. This set of relation trees is always the set of bases of a matroid (Proposition 2.4), which in case of a generic ideal consists of only one tree as shown in Proposition 2.1. We call the graph  $G$  whose set of edges is the union of the set of edges of all relation trees of a given Cohen-Macaulay monomial ideal  $I$  of codimension 2, the Taylor graph of  $I$ . Then each of the relation trees is a spanning tree of the Taylor graph. The natural question arises whether the set of relation trees of  $I$  is precisely the set of spanning trees of  $G$ . We show by an example that this is not the case in general. On the other hand, we prove in Theorem 2.5 that each relation tree of  $I$  is a spanning tree of  $G$ , if  $I$  has a linear resolution. In order to obtain a complete description of all possible relation trees when  $I$  has a linear resolution, it is therefore required to find all possible Taylor graphs of such ideals. This is done in Theorem 2.6, where it is shown that a finite connected simple graph is the Taylor graph of Cohen-Macaulay monomial ideal of codimension 2 with linear resolution, if and only if  $G$  is chordal and any two maximal cliques of  $G$  have at most one vertex in common.

# 1. ON THE STRUCTURE OF COHEN-MACAULAY MONOMIAL IDEALS OF CODIMENSION 2

In [1, Remark 6.3] the following observation was made regarding the structure of a codimension 2 Cohen-Macaulay monomial ideal  $I$ : let

$$\{u_1, u_2, \dots, u_{m+1}\}$$

the unique minimal set of monomial generators of  $I$ . Consider the Taylor complex of the sequence  $u_1, u_2, \dots, u_{m+1}$

$$\dots \rightarrow \bigoplus_{i=1}^{\binom{m+1}{2}} S e_i \wedge e_j \xrightarrow{\varphi_2} \bigoplus_{i=1}^{m+1} S e_i \xrightarrow{\varphi_1} S$$

The matrix corresponding to  $\varphi_2$  is of size  $\binom{m+1}{2} \times m+1$  whose rows correspond to Taylor relation (cf. [4]), namely to the relations

$$e_i \wedge e_j \mapsto u_{ji}e_j - u_{ij}e_i$$

where  $i < j$  and  $u_{ji} = u_i / \gcd(u_i, u_j)$ ,  $u_{ij} = u_j / \gcd(u_i, u_j)$ .

Let  $U = \text{Ker}(\varphi_1)$ ; then the Taylor relations form a homogeneous system of generators of  $U$ . Since  $\text{projdim } S/I = 2$ , it follows that  $U$  is free of rank  $m$ . In particular  $U$  is minimally generated by  $m$  elements. Applying the graded Nakayama Lemma (cf. [2] or [6, Lemma 1.2.6]), a minimal system of graded generators of  $U$  can be chosen among the Taylor relations. We then obtain a minimal graded free resolution

$$0 \rightarrow S^m \xrightarrow{A} S^{m+1} \rightarrow S \rightarrow S/I \rightarrow 0$$

of  $S/I$ , where  $A$  is a matrix whose rows correspond to Taylor relations. Any such matrix will be called a *Hilbert–Burch matrix* of  $I$ .

Notice that each row of  $A$  has exactly two nonzero entries. We obtain a graph  $\Gamma$  on the vertex set  $[m+1] = \{1, \dots, m+1\}$  from the matrix  $A$  as follows: we say that  $\{i, j\}$  is an edge of  $\Gamma$ , if and only if there is a row of  $A$  whose nonzero entries are the  $i$ th and  $j$ th components.

We claim that every column of  $A$  has a nonzero entry. In fact, if this would not be the case, say, the  $k$ th column of  $A$  has all entries zero, then the relation  $u_{k+1,k}e_{k+1} - u_{k,k+1}e_k \in U$  could not be written as a linear combination of the minimal graded homogeneous generators of  $U$ . This shows that  $\Gamma$  has no isolated vertex. On the other hand, since the number of vertices of  $\Gamma$  is  $m+1$  and the number of edges of  $\Gamma$  is  $m$ , we see that  $\Gamma$  is a tree, which is called a *relation tree* of  $I$ . The set of all relation trees of  $I$  will be denoted by  $\mathcal{T}(I)$ .

Conversely, given a tree  $\Gamma$  on the vertex set  $[m+1]$  with  $m \geq 2$ , we are going to construct a codimension 2 Cohen-Macaulay monomial ideal  $I$  for which  $\Gamma$  is a relation tree. We assign to  $\Gamma$  an  $m \times (m+1)$ -matrix  $A(\Gamma) = (a_{ij})$  whose entries are either 0 or indeterminates. The matrix  $A(\Gamma)$  is defined as follows: let  $E(\Gamma)$  be the set of edges of  $\Gamma$ . Since  $\Gamma$  is a tree, there are exactly  $m$  edges. We choose an arbitrary order of the edges of

$\Gamma$ , and assign to the  $k$ th edge  $\{i, j\} \in E(\Gamma)$  the  $k$ th row of  $A(\Gamma)$  by

$$(1) \quad a_{kl} = \begin{cases} -x_{ij} & \text{if } l = i, \\ x_{ji} & \text{if } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

For example if  $\Gamma$  is the tree with edges  $\{1, 2\}$ ,  $\{2, 3\}$  and  $\{2, 4\}$ . Then we obtain the matrix

$$A(\Gamma) = \begin{pmatrix} -x_{12} & x_{21} & 0 & 0 \\ 0 & -x_{23} & x_{32} & 0 \\ 0 & -x_{24} & 0 & x_{42} \end{pmatrix}$$

**Definition 1.1.** Let  $\Gamma$  be a tree on the vertex set  $[m+1]$  and  $i, j$  be two distinct vertices of  $\Gamma$ . Then there exists a unique path from  $i$  to  $j$  denoted by  $i \rightarrow j$ , in other words a sequence of numbers  $i = i_0, i_1, i_2, \dots, i_{k-1}, i_k = j$  such that  $\{i_l, i_{l+1}\} \in E(\Gamma)$  for  $l = 0, \dots, k-1$ . We set

$$b(i, j) = i_1 \quad \text{and} \quad e(i, j) = i_{k-1}$$

**Proposition 1.2.** Let  $v_j$  be the minor of  $A(\Gamma)$  which is obtained by omitting the  $j$ th column of  $A(\Gamma)$ . Then  $v_j = \pm \prod_{\substack{i=1 \\ i \neq j}}^{m+1} x_{ib(i,j)}$  for  $j = 1, 2, \dots, m+1$

*Proof.* We prove the assertion by using induction on the number of edges of  $\Gamma$ . If  $|E(\Gamma)| = 1$ , then

$$A(\Gamma) = (-x_{12}, x_{21})$$

Therefore,  $v_1 = x_{21}$  and  $v_2 = -x_{12}$ , as required.

Now assume that the assertion is true for  $|E(\Gamma)| = m-1 \geq 1$ . Since  $\Gamma$  is a tree, there exists a free vertex of  $\Gamma$ , that is, a vertex which belongs to exactly one edge. Such an edge of  $\Gamma$  is called a leaf. We may assume the edge  $\{m, m+1\}$  is a leaf and that  $m+1$  is a free vertex of  $\Gamma$ . The tree which is obtained from  $\Gamma$  by removing the leaf  $\{m, m+1\}$  will be denoted by  $\Gamma'$ . By our induction hypothesis the minors  $v'_1, \dots, v'_m$  of  $\Gamma'$  have the desired form. We may assume that the edge  $\{m, m+1\}$  is the last in the order of edges. Then  $(m-1) \times m$  matrix  $A(\Gamma')$  is obtained from the  $m \times (m+1)$ -matrix  $A(\Gamma)$  by removing the last row

$$R_m = (0, \dots, 0, -x_{m,m+1}, x_{m+1,m})$$

and the last column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{m+1,m} \end{pmatrix}$$

It follows that the minors  $v_1, \dots, v_{m+1}$  of  $A(\Gamma)$  are given by

$$(2) \quad v_j = x_{m+1,m} v'_j \quad \text{for } j = 1, \dots, m, \quad \text{and} \quad v_{m+1} = x_{m,m+1} v'_m.$$

Therefore, our induction hypothesis implies that

$$v_j = x_{m+1,m} v'_j = \pm x_{m+1,m} \prod_{\substack{i=1 \\ i \neq j}}^m x_{i,b(i,j)} = \pm \prod_{\substack{i=1 \\ i \neq j}}^{m+1} x_{i,b(i,j)}$$

for  $j = 1, \dots, m$ , and

$$v_{m+1} = x_{m,m+1}v'_m = \pm x_{m,m+1} \prod_{i=1}^{m-1} x_{i,b(i,m)} = \pm x_{m,b(m,m+1)} \prod_{i \neq 1}^{m-1} x_{i,b(i,m+1)},$$

because  $b(i, m) = b(i, m+1)$  for all  $i \leq m$ . So this implies that

$$v_{m+1} = \pm \prod_{i=1, i \neq m+1}^{m+1} x_{i,b(i,m+1)},$$

as desired.  $\square$

For a tree  $\Gamma$  on the vertex set  $[m+1]$  we denote by  $I(\Gamma)$  the ideal generated by the minors  $v_1, \dots, v_{m+1}$  of  $A(\Gamma)$  and call it the *generic monomial ideal* attached to the tree  $\Gamma$ .

**Corollary 1.3.** *The ideal  $I(\Gamma)$  is a Cohen–Macaulay ideal of codimension 2.*

*Proof.* The greatest common divisors of the monomial generators  $v_j$  of  $I(\Gamma)$  is one. This can easily be seen by the formulas (2) in the proof of Proposition 1.2. The assertion follows then from [2, Theorem 1.4.17].  $\square$

The generic ideal  $I(\Gamma)$  has the following nice primary decomposition:

**Proposition 1.4.**  $I(\Gamma) = \bigcap_{1 \leq i < j \leq m+1} (x_{ib(i,j)}, x_{je(i,j)}).$

*Proof.* We prove the assertion by using induction on the number of edges of  $\Gamma$ . For  $|E(\Gamma)| = 1$  we have,

$$A(\Gamma) = (-x_{12}, x_{21}).$$

with  $v_1 = x_{21}$ ,  $v_2 = -x_{12}$ . Therefore  $I(\Gamma) = (x_{21}, x_{12}) = (x_{1b(1,2)}, x_{2e(1,2)})$ . Now assume that assertion is true if  $|E(\Gamma)| = m-1 \geq 1$ . Since  $\Gamma$  is a tree, there exists a free vertex of  $\Gamma$ , that is, a vertex which belongs to exactly one edge. Such an edge of  $\Gamma$  is called a leaf. We may assume the  $\{m, m+1\}$  is a leaf and that  $m+1$  is a free vertex of  $\Gamma$ . The tree which is obtained from  $\Gamma$  by removing the leaf  $\{m, m+1\}$  will be denoted by  $\Gamma'$ . So then for  $A(\Gamma')$  we have

$$I(\Gamma') = (v'_1, v'_2, \dots, v'_m) = \bigcap_{1 \leq i < j \leq m} (x_{ib(i,j)}, x_{je(i,j)}).$$

We may assume that the edge  $\{m, m+1\}$  is the last in the order of edges. Then  $(m-1) \times m$  matrix  $A(\Gamma')$  is obtained from the  $m \times (m+1)$ -matrix  $A(\Gamma)$  by deleting the last row

$$R_m = (0, \dots, 0, -x_{m,m+1}, x_{m+1,m})$$

and the last column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{m+1,m} \end{pmatrix}$$

It follows that the minors  $v_1, \dots, v_{m+1}$  of  $A(\Gamma)$  are given by

$$v_j = x_{m+1,m}v'_j \quad \text{for } j = 1, \dots, m, \quad \text{and} \quad v_{m+1} = x_{m,m+1}v'_m.$$

Hence

$$I(\Gamma) = (v_1, v_2, \dots, v_{m+1}).$$

On the other hand, by using the induction hypothesis and the fact that  $e(i, m+1) = m$  for all  $i \leq m$ , we get

$$\begin{aligned} \bigcap_{1 \leq i < j \leq m+1} (x_{ib(i,j)}, x_{je(i,j)}) &= \bigcap_{1 \leq i < j \leq m} (x_{ib(i,j)}, x_{je(i,j)}) \cap \bigcap_{i=1}^m (x_{ib(i,m+1)}, x_{m+1,e(i,m+1)}) \\ &= (v'_1, v'_2, \dots, v'_m) \cap \bigcap_{i=1}^m (x_{ib(i,m+1)}, x_{m+1,m}) \\ &= (v'_1, v'_2, \dots, v'_m) \cap \left( \prod_{i=1}^m x_{ib(i,m+1)}, x_{m+1,m} \right) \\ &= (v'_1, v'_2, \dots, v'_m) \cap \left( x_{m,m+1} \prod_{i=1}^{m-1} x_{ib(i,m+1)}, x_{m+1,m} \right) \\ &= (v'_1, v'_2, \dots, v'_m) \cap (x_{m,m+1} v'_m, x_{m+1,m}). \end{aligned}$$

Observing that  $\gcd(v'_i, x_{m+1,m}) = 1$  it follows that

$$\begin{aligned} (v'_1, v'_2, \dots, v'_m) \cap (x_{m,m+1} v'_m, x_{m+1,m}) &= (x_{m+1,m} v'_1, x_{m+1,m} v'_2, \dots, x_{m+1,m} v'_m, x_{m,m+1} v'_m) \\ &= (v_1, v_2, \dots, v_m, v_{m+1}) = I(\Gamma). \end{aligned}$$

Hence

$$I(\Gamma) = \bigcap_{1 \leq i < j \leq m+1} (x_{ib(i,j)}, x_{je(i,j)}),$$

as desired.  $\square$

As an application of Proposition 1.2, Corollary 1.3 and Proposition 1.4 we obtain the following characterization of Cohen-Macaulay monomial ideals of codimension 2.

**Theorem 1.5.** (a) *Let  $I \subset S = K[x_1, x_2, \dots, x_n]$  be a Cohen-Macaulay monomial ideal of codimension 2 generated by  $m+1$  elements. Then there exists a tree  $\Gamma$  with  $m+1$  vertices and for each edge  $\{i, j\}$  of  $\Gamma$  there exists a monomials  $u_{ij}$  and  $u_{ji}$  in  $S$  such that*

- (i)  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$  for all  $i < j$ , and
- (ii)  $I = (\prod_{i=2}^{m+1} u_{ib(i,1)}, \dots, \prod_{\substack{i=1 \\ i \neq j}}^{m+1} u_{ib(i,j)}, \dots, \prod_{i=1}^m u_{ib(i,m+1)})$

(b) *Conversely, if  $\Gamma$  is a tree with  $[m+1]$  vertices and for each  $\{i, j\} \in E(\Gamma)$  we are given monomials  $u_{ij}$  and  $u_{ji}$  in  $S$  satisfying (a)(i). Then the ideal defined in (a)(ii) is Cohen-Macaulay of codimension 2.*

*Proof.* (a) (ii) Let  $A$  be an  $m \times m+1$  matrix of Taylor relations which generated the relation module of  $U$  of  $I$ , and let  $\Gamma$  be the corresponding relation tree. We apply the Hilbert–Burch Theorem ([2, 1.4.17]) according to which the ideal  $I$  is generated by the maximal minors of  $A$ . The matrix  $A$  is obtained from  $A(\Gamma)$  by the substitution:

$$x_{ij} \mapsto u_{ij}.$$

Therefore statement (ii) follows from Proposition 1.2.

Now we shall prove assertion (i). For this we use Proposition 1.4 which says that

$$I(\Gamma) = \bigcap_{1 \leq i < j \leq m+1} (x_{ib(i,j)}, x_{je(i,j)}).$$

Applying the substitution map introduced in the proof of (ii) we obtain

$$(3) \quad I \subseteq \bigcap_{i < j} (u_{ib(i,j)}, u_{je(i,j)}).$$

Suppose  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) \neq 1$  for some  $i$  and  $j$ . Then it follows from (3) that  $I$  is contained in a principal ideal. This is a contradiction, because height  $I = 2$ .

(b) Let  $\Gamma$  be a tree with vertex set  $[m+1]$  and  $m$  edges. For each  $\{i, j\} \in E(\Gamma)$  we have monomials  $u_{ij}, u_{ji} \in S$  satisfying condition (a)(i). Let  $A$  be the matrix obtained from  $A(\Gamma)$  by the substitutions  $x_{ij} \mapsto u_{ij}$ , and let  $I$  be the ideal generated by the maximal minors of  $A$ . It follows from Proposition 1.2 that  $I = (v_1, \dots, v_{m+1})$  where  $v_j = \prod_{\substack{i=1 \\ i \neq j}}^{m+1} u_{ib(i,j)}$ .

First we shall prove that

$$\gcd(v_1, v_2, \dots, v_{m+1}) = 1.$$

We shall prove this by induction on the number of edges of  $\Gamma$ . The assertion is trivial if  $\Gamma$  has only one edge. Now let  $|E(\Gamma)| = m > 1$  and assume that the assertion is true for any tree with  $m-1$  edges.

We may assume that  $(m, m+1)$  is a leaf of  $\Gamma$ . Let  $\Gamma'$  be the tree obtained from  $\Gamma$  by removing the edge  $\{m, m+1\}$ . The matrix  $A(\Gamma')$  is obtained from  $A(\Gamma)$  by removing the row  $(0, \dots, -x_{m,m+1}, x_{m+1,m})$  and the column

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{m+1,m} \end{pmatrix}.$$

Let  $A'$  be the matrix obtained from  $A(\Gamma')$  by the substitutions  $x_{ij} \mapsto u_{ij}$ , and let  $I' = (v'_1, \dots, v'_m)$  be the ideal of maximal minors of  $A'$  where, up to sign,  $v'_j$  is the  $j$ th maximal minor of  $A'$ . Expanding the matrix  $A$  we see that

$$v_j = \pm v'_j u_{m+1,m} \quad \text{for } j = 1, 2, \dots, m \quad \text{and} \quad v_{m+1} = \pm v'_m u_{m,m+1}.$$

Therefore

$$\gcd(v_1, v_2, \dots, v_m, v_{m+1}) = \gcd(v'_1 u_{m+1,m}, v'_2 u_{m+1,m}, \dots, v'_m u_{m+1,m}, v_{m+1}).$$

By induction hypothesis we have  $\gcd(v'_1, v'_2, \dots, v'_m) = 1$ , so that

$$\gcd(v'_1 u_{m+1,m}, v'_2 u_{m+1,m}, \dots, v'_m u_{m+1,m}) = u_{m+1,m}.$$

Hence it is enough to prove that

$$\gcd(u_{m+1,m}, v_{m+1}) = 1.$$

Note that  $u_{m+1,m} = u_{m+1,e(i,m+1)}$  for all  $i$ , and  $v_{m+1} = \prod_{i=1}^m u_{ib(i,m+1)}$ . Therefore

$$\gcd(u_{m+1,m}, v_{m+1}) = \gcd(u_{m+1,e(i,m+1)}, \prod_{i=1}^m u_{ib(i,m+1)}) = 1,$$

since by our hypothesis (a)(i) we have  $\gcd(u_{m+1,e(i,m+1)}, u_{ib(i,m+1)}) = 1$  for all  $i$ .

The Hilbert–Burch Theorem [2, 1.4.17] then implies that  $I$  is a perfect ideal of codimension 2, and hence a Cohen–Macaulay ideal.  $\square$

## 2. THE POSSIBLE SETS OF RELATION TREES ATTACHED TO COHEN–MACAULAY MONOMIAL IDEALS OF CODIMENSION 2

In this section we want to study set  $\mathcal{T}(I)$  of all relation trees of a Cohen–Macaulay monomial ideal of codimension 2. In general one may have more than just one Hilbert–Burch matrix for an ideal  $I$ , and consequently more than one relation trees. For example the ideal  $I = (x_4x_5x_6, x_1x_5x_6, x_1x_2x_6, x_1x_2x_3x_5) \subset S = K[x_1, x_2, x_3, x_4, x_5, x_6]$  has the following two Hilbert–Burch matrices

$$A_1 = \begin{pmatrix} -x_1 & x_4 & 0 & 0 \\ 0 & -x_2 & x_5 & 0 \\ 0 & 0 & -x_3x_5 & x_6 \end{pmatrix},$$

or

$$A_2 = \begin{pmatrix} -x_1 & x_4 & 0 & 0 \\ 0 & -x_2 & x_5 & 0 \\ 0 & -x_2x_3 & 0 & x_6 \end{pmatrix}.$$

The corresponding relation trees are  $\Gamma_1$  and  $\Gamma_2$  with  $E(\Gamma_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$  and  $E(\Gamma_2) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$ .

However in the generic case we have

**Proposition 2.1.** *Let  $\Gamma$  be a tree on the vertex set  $[m+1]$  and let  $I(\Gamma)$  be the generic monomial ideal attached to  $\Gamma$ . Then  $\mathcal{T}(I(\Gamma)) = \{\Gamma\}$ .*

Recall that  $I(\Gamma)$  is the ideal of maximal minors of the matrix  $A(\Gamma)$  defined in (1). Up to signs the minors of  $A(\Gamma)$  are the monomials  $v_i = \prod_{\substack{r=1 \\ r \neq i}}^{m+1} x_{rb(r,i)}$ , see Proposition 1.2.

For the proof of Proposition 2.1 we shall need

**Lemma 2.2.** *Let  $\Gamma$  be a tree, then  $\{i, j\}$  is an edge of  $\Gamma$  if and only if*

$$\text{lcm}(v_i, v_j) = v_j x_{ji} = v_i x_{ij}.$$

*Proof.* Let  $\{i, j\}$  be an edge of  $\Gamma$  and suppose that  $i < j$ . Note that

$$(4) \quad b(k, i) = b(k, j)$$

for all  $k$  which are different from  $i$  and  $j$ , because if the path from  $k$  to  $i$  is  $k = k_0, k_1, \dots, k_l = i$ , then the path from  $k$  to  $j$  will be  $k = k_0, k_1, \dots, k_{l-1} = j$  or  $k =$

$k_0, k_1, \dots, k_{l-1}, i, j$  since  $\{i, j\}$  be an edge of  $\Gamma$ . Now using (4) we have

$$v_i = \pm \prod_{\substack{r=1 \\ r \neq i}}^{m+1} x_{rb(r,i)} = \pm \prod_{\substack{r=1 \\ r \neq i, j}}^{m+1} x_{rb(r,i)} x_{jb(j,i)} = \pm \prod_{\substack{r=1 \\ r \neq i, j}}^{m+1} x_{rb(r,i)} x_{ji} = \pm \prod_{\substack{r=1 \\ r \neq i, j}}^{m+1} x_{rb(r,j)} x_{ji}.$$

Similarly  $v_j = \pm \prod_{\substack{r=1 \\ r \neq i, j}}^{m+1} x_{rb(r,j)} x_{ij}$ . Hence  $\text{lcm}(v_i, v_j) = v_j x_{ji} = v_i x_{ij}$ .

On the other hand, suppose that  $\{i, j\}$  is not an edge of  $\Gamma$ , then there exists a vertex, different from  $i$  and  $j$ , say  $k$ , which belongs to the path from  $i$  to  $j$ . Therefore  $b(k, i) \neq b(k, j)$ , and hence  $x_{kb(k,i)} \neq x_{kb(k,j)}$ . Since  $x_{kb(k,i)} \mid v_i$  and since  $x_{kb(k,j)} \mid v_j$  we cannot have  $\text{lcm}(v_i, v_j) = v_j x_{ji} = v_i x_{ij}$ .  $\square$

*Proof of Proposition 2.1.* Since all monomial generators of  $I(\Gamma)$  are of degree  $m$  and since, by the Hilbert–Burch Theorem [2, 1.4.17], these generators are the maximal minors of any of its Hilbert–Burch matrices, it follows that all Hilbert–Burch matrices must be linear. However by Lemma 2.2 we have only  $m$  linear Taylor relations. Therefore there exists only one Hilbert–Burch matrix for  $I$ .  $\square$

In contrast to the result stated in Proposition 2.1 we have

**Proposition 2.3.** *Let  $I = (u_1, \dots, u_{m+1})$  be the monomial ideal in  $K[x_1, \dots, x_{m+1}]$  with  $u_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_{m+1}$  for  $i = 1, \dots, m+1$ . Then  $\mathcal{T}(I)$  is the set of all possible trees on the vertex set  $[m+1]$ .*

*Proof.* Let  $\Gamma$  be an arbitrary tree on the vertex set  $[m+1]$ . For the  $k$ th edge  $\{i, j\}$  of  $\Gamma$  take the monomial generators  $u_i$  and  $u_j$  of  $I$ . Then we have the Taylor relation  $x_j e_j - x_i e_i$ . Let  $A$  be the  $m \times m+1$ -matrix whose rows  $(0, \dots, -x_i, \dots, x_j, \dots, 0)$  correspond to the Taylor relations  $x_j e_j - x_i e_i$  arising from the edges of  $\Gamma$ . Observe that the generic matrix  $A(\Gamma)$  is mapped to  $A$  by the substitutions  $x_{ij} = x_i$ . Moreover the maximal minor  $\pm v_i$  of  $A(\Gamma)$  is mapped to  $u_i$  for all  $i$ . Therefore the  $u_i$  are the maximal minors of  $A$  which shows that  $A$  is the Hilbert–Burch matrix of  $I$ .  $\square$

In order to study the general nature of  $\mathcal{T}(I)$  we introduce the following concept. Let  $\mathcal{S}$  be a finite set. Recall that a collection  $\mathcal{B}$  of subsets of  $\mathcal{S}$  is said to be the *set of bases of a matroid*, if all  $B \in \mathcal{B}$  have the same cardinality and if the following exchange property is satisfied:

For all  $B_1, B_2 \in \mathcal{B}$  and  $i \in B_1 \setminus B_2$ , there exists  $j \in B_2 \setminus B_1$  such that  $(B_1 \setminus \{i\}) \cup \{j\} \in \mathcal{B}$ .

A classical example is the following: let  $K$  be a field,  $V$  a  $K$ -vector space and  $\mathcal{S} = \{v_1, \dots, v_r\}$  any finite set of vectors of  $V$ . Let  $\mathcal{B}$  the set of subset  $B$  of  $\mathcal{S}$  with the property that  $B$  is a maximal set of linearly independent vectors in  $\mathcal{S}$ . It easy to check and well known that  $\mathcal{B}$  is the set of bases of a matroid.

**Proposition 2.4.** *Let  $I \subset S$  be a Cohen–Macaulay monomial ideal of codimension 2. Then  $\mathcal{T}(I)$  is the set of bases of a matroid.*

*Proof.* Let  $I$  be minimally generated by the monomials  $u_1, \dots, u_{m+1}$  and let

$$0 \longrightarrow G \longrightarrow F \longrightarrow I \longrightarrow 0$$



be the graded minimal free  $S$ -resolution of  $S/I$ .

The set  $\mathcal{S}$  of Taylor relations generate the first syzygy module  $U$  of  $I$  which is isomorphic to the free  $S$ -module  $G$ . Consider the graded  $K$ -vector space  $U/\mathfrak{m}U$  where  $\mathfrak{m} = (x_1, \dots, x_n)$  is the graded maximal ideal of  $S$ . Note that  $\dim_K U/\mathfrak{m}U = m$ . Since the relations  $r_{ij}$  generate  $U$  it follows that their residue classes  $\bar{r}_{ij}$  in the  $K$ -vector space  $U/\mathfrak{m}U$  form a system of generators of  $U/\mathfrak{m}U$ . By the homogeneous version of Nakayama (see [2, 1.5.24]) it follows that a subset  $B = \{r_{i_1 j_1}, \dots, r_{i_m j_m}\}$  of the Taylor relations  $\mathcal{S}$  is a minimal set of generators of  $U$  (and hence establishes a Hilbert–Burch matrix of  $I$ ) if and only if  $\{\bar{r}_{i_1 j_1}, \dots, \bar{r}_{i_m j_m}\}$  is a basis of the  $K$ -vector space  $U/\mathfrak{m}U$ . The desired conclusion follows, since the relation trees of  $I$  correspond bijectively to the set of Hilbert–Burch matrices of  $I$ .  $\square$

Given a finite simple and connected graph  $G$ . A maximal subtree  $\Gamma \subset G$  is called a *spanning tree*. It is well-known and easy to see that the set  $\mathcal{T}(G)$  of spanning trees is the set of bases of a matroid.

Here we are interested in the spanning trees of the graph  $G(I)$  whose set of edges is given by with

$$E(G(I)) = \bigcup_{\Gamma \in \mathcal{T}(I)} E(\Gamma).$$

We call  $G(I)$  the *Taylor graph* of  $I$ . Obviously we have  $\mathcal{T}(I) \subset \mathcal{T}(G(I))$ . The question arises whether  $\mathcal{T}(I) = \mathcal{T}(G(I))$ ? Unfortunately this is not always the case as the example at the beginning of this section shows. Indeed, in this example,  $\mathcal{T}(I) = \{\Gamma_1, \Gamma_2\}$  with  $E(\Gamma_1) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$  and  $E(\Gamma_2) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$ , so that  $E(G_I) = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ . This graph has the spanning trees  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$  with  $E(\Gamma_3) = \{\{1, 2\}, \{2, 4\}, \{3, 4\}\}$ . If  $\Gamma_3$  would be a relation tree of  $I$ , then

$$A = \begin{pmatrix} -x_1 & x_4 & 0 & 0 \\ 0 & -x_2 x_3 & 0 & x_6 \\ 0 & 0 & -x_3 x_5 & x_6 \end{pmatrix}.$$

would have to be a Hilbert–Burch matrix of  $I$ , which is not the case since the ideal of maximal minors of  $A$  is the ideal  $x_3 I$ .

However we have

**Theorem 2.5.** *Let  $I$  be Cohen–Macaulay monomial ideal of codimension 2 with linear resolution. Then  $\mathcal{T}(I) = \mathcal{T}(G(I))$ .*

*Proof.* Since  $I$  has a linear resolution, it follows that all Hilbert–Burch matrices of  $I$  are matrices with linear entries. Let  $L = \{r_1, \dots, r_k\}$  be the set of linear Taylor relations. We may assume that  $r_1, \dots, r_m$  are the rows of a Hilbert–Burch matrix of  $I$ , in other words, that  $r_1, \dots, r_m$  is a basis of the first syzygy module  $U$  of  $I$ .

We first claim that  $r_{i_1}, \dots, r_{i_m} \in L$  is basis of  $U$  if and only if the relations  $r_{i_1}, \dots, r_{i_m}$  are  $K$ -linear independent. Obviously, the relations must be  $K$ -linear independent in order to form a basis of the free  $S$ -module  $U$ . Conversely, assume that  $r_{i_1}, \dots, r_{i_m}$  are  $K$ -linear independent. Since each  $r_{i_j}$  belongs to  $U$  we can write

$$r_{i_j} = f_{1j}r_1 + f_{2j}r_2 + \dots + f_{mj}r_m \quad \text{with} \quad f_{lj} \in S.$$

The presentation can be chosen such that all  $f_{lj}$  are homogeneous and such that  $\deg f_{lj}r_l = \deg r_{i_j} = 1$  for all  $l$  and  $j$ . In other words,  $\deg f_{lj} = 0$  for all  $l$  and  $j$ . Therefore the  $m \times m$ -matrix  $F = (f_{lj})$  is a matrix with coefficients in  $K$ . Since, by assumption the relations  $r_{i_1}, \dots, r_{i_m}$  are  $K$ -linear independent, it follows that  $F$  is invertible. This implies that the relations  $r_1, \dots, r_m$  are linear combinations of the relations  $r_{i_1}, \dots, r_{i_m}$ . Therefore these relations generate  $U$  as well, and in fact form a basis of  $U$ , since  $U$  is free of rank  $m$ .

Our considerations so far have shown, that the set of Hilbert–Burch matrices of  $I$  correspond bijectively to the maximal  $K$ -linear subsets of  $L$ . Each  $r_i \in L$  is a row vector with exactly two non-zero entries. We attach to  $r_i$  the edge  $e_i = \{k, l\}$ , if the two non-zero entries of  $r_i$  are at position  $k$  and  $l$ , and claim that

$$E(G(I)) = \{e_1, \dots, e_k\}.$$

Indeed, according to the definition of  $G(I)$  an edge  $e$  belongs to  $E(G(I))$ , if there exists a relation tree  $T$  of  $I$  with  $e \in E(T)$ . This is equivalent to say that there exist linearly independent  $r_{i_1}, \dots, r_{i_m} \in L$  such that  $e = e_{i_j}$  for some  $j$ . Now choose  $e_i \in \{e_1, \dots, e_k\}$ . Then  $r_i$  can be completed to maximal set  $\{r_i, r_{i_2}, \dots, r_{i_m}\}$  of  $K$ -linear elements in  $L$ . This shows that  $e_i \in E(G(I))$  for  $i = 1, \dots, k$ , so that  $\{e_1, \dots, e_k\} \subset E(G(I))$ . The other inclusion is trivially true.

In order to complete the proof of the theorem we need to show that each spanning tree  $T$  of  $G(I)$  is a relation tree of  $I$ . Let  $e_{i_1}, \dots, e_{i_m}$  be the edges of the tree. To prove that  $T$  is a relation tree amounts to show the relations  $r_{i_1}, \dots, r_{i_m}$  are  $K$ -linearly independent.

A free vertex of  $T$  is a vertex which belongs to exactly one edge. Since  $T$  is a tree, it has at least one free vertex. Say, 1 is this vertex and  $e_{i_1}$  is the edge to which the free vertex 1 belongs. Removing the edge  $e_{i_1}$  from  $T$  we obtain a tree  $T'$  on the vertex set  $\{2, 3, \dots, m+1\}$ . After renumbering the vertices and edges if necessary, we may assume that 2 is a free vertex of  $T'$  and  $e_{i_2}$  the edge to which 2 belongs. Proceeding in this way we get, after a suitable renumbering of the vertices and edges of  $T$ , a free vertex ordering of the edges, that is, for all  $j = 1, \dots, r$  the edges  $e_{i_j}, e_{i_{j+1}}, \dots, e_{i_m}$  is the set of edges of a tree for which  $j$  is a free vertex belonging to  $e_{i_j}$ . Since renumbering of vertices and of edges of  $T$  means for the corresponding matrix of relations simply permutation of the rows and columns, the rank of relation matrix is unchanged. However in this new ordering, if we skip the last column of the  $m \times m+1$  relation matrix we obtain an upper triangular  $m \times m$  matrix with non-zero entries on the diagonal. This shows that the relations  $r_{i_1}, \dots, r_{i_m}$  are  $K$ -linearly independent, as desired.  $\square$

Finally we will describe all the possible Taylor graphs of a Cohen–Macaulay monomial ideal of codimension 2 with linear resolution. Then, together with Theorem 2.5, we have a complete description of all possible relation trees for such ideals.

Let  $G$  be finite connected simple graph on the vertex set  $[n]$ . Recall that a subset  $C$  of  $[n]$  is called a *clique* of  $G$  if for all  $i$  and  $j$  belonging to  $C$  with  $i \neq j$  one has  $\{i, j\} \in E(G)$ . The set of all cliques  $\Delta(G)$  is a simplicial complex, called the *clique complex* of  $G$ .

**Theorem 2.6.** *Let  $G$  be finite connected simple graph. Then the following are equivalent:*

- (a)  *$G$  is a Taylor graph of a Cohen–Macaulay monomial ideal of codimension 2 with linear resolution.*

- (b)  $G$  is a chordal graph with the property that any two distinct maximal cliques have at most one vertex in common.

*Proof.* (a)  $\Rightarrow$  (b): Let  $I$  be generated by  $m$  monomials and  $G = G(I)$ , and let  $C$  be a cycle of  $G$ . We first show that the restriction  $G'$  of  $G$  to  $C$  is a complete graph, that is, we show that for any two distinct vertices  $i, j \in C$  it follows that  $\{i, j\} \in E(G)$ . In particular, this will imply that  $G$  is chordal.

For simplicity we may assume that  $E(C) = \{e_1, \dots, e_k\}$  with  $k \geq 3$  and  $e_i = \{i, i+1\}$  for  $i = 1, \dots, k-1$  and  $e_k = \{k, 1\}$ . Let  $r_1, \dots, r_k$  be the corresponding relations. Let  $\varepsilon_i \in K^{m-1}$ ,  $i = 1, \dots, m-1$  be the canonical basis vectors of  $K^{m-1}$ . Then  $r_i = -a_i\varepsilon_i + b_i\varepsilon_{i+1}$  for  $i = 1, \dots, k-1$  and  $r_k = -b_k\varepsilon_1 + a_k\varepsilon_k$ , where  $a_i$  and  $b_i$  belong to  $\{x_1, \dots, x_n\}$ . Assume that  $r_1, \dots, r_k$  are  $K$ -linearly independent. Then  $r_1, \dots, r_k$  can be completed to  $K$ -basis  $r_1, \dots, r_m$  of  $L$ . (Here we use the notation introduced in the proof of Theorem 2.5.) Let  $\Gamma$  be the tree corresponding to  $r_1, \dots, r_m$ . Then  $C$  is a subgraph of  $\Gamma$ , which is a contradiction. Thus we see that the relations  $r_1, \dots, r_k$  are  $K$ -linearly dependent which implies at once that  $a_1 = b_k$  and  $a_i = b_{i-1}$  for  $i = 2, \dots, k$ . Hence we have  $r_1 + \dots + r_i = -a_1\varepsilon_1 + b_i\varepsilon_{i+1}$  for  $i = 1, \dots, k-1$ . This implies that  $\{1, i\}$  is an edge of  $G$  for  $i = 2, \dots, k$ . By symmetry, also the other edges  $\{i, j\}$  with  $2 \leq i < j \leq k$  belong to  $G$ .

Now let  $G_1$  and  $G_2$  be two distinct maximal cliques of  $G$ , and assume that they have two vertices in common, say, the vertices  $i$  and  $j$ . Let  $k \in G_1 \setminus \{i, j\}$  and  $l \in G_2 \setminus \{i, j\}$ . Then the graph  $C$  with edges  $\{i, k\}, \{k, j\}, \{j, l\}, \{l, i\}$  is a cycle in  $G$ . Therefore, by what we have shown, it follows that  $\{k, l\}$  is an edge of  $G$ . Thus for any two vertices  $k, l \in V(G_1) \cup V(G_2)$  it follows that  $\{k, l\} \in E(G)$ , contradicting the fact that  $G_1$  and  $G_2$  are distinct maximal cliques of  $G$ .

(b)  $\Rightarrow$  (a): Let  $C_1, \dots, C_r$  be the maximal cliques of the chordal graph  $G$ , and let  $\Delta(G)$  be the clique complex of  $G$ . Then the  $C_i$  are the facets of  $\Delta(G)$ . One version of Dirac's theorem [3] says that  $\Delta(G)$  is a quasi-forest, see [5]. This means, that there is an order of the facets, say,  $C_1, C_2, \dots, C_r$  such that for each  $i$  there is a  $j < i$  with the property that  $C_k \cap C_i \subset C_j \cap C_i$  for all  $k < i$ . Given this order, then our hypothesis (b) implies that for each  $i = 2, \dots, r$  there exists a vertex  $k_i \in C_i$  such  $C_i \cap C_{i-1} = \{k_i\}$  and  $C_i \cap C_j = \{k_i\}$  for all  $j < i$  with  $C_i \cap C_j \neq \emptyset$ . The following example illustrates the situation. Let  $G$  be the graph on the vertex set [7] with edges  $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{5, 6\}, \{5, 7\}$ .

Then  $G$  is a connected simple graph satisfying the condition in (b). The maximal cliques of  $G$  ordered as above are  $C_1 = \{1, 2, 3\}$ ,  $C_2 = \{3, 4, 5\}$ ,  $C_3 = \{5, 6\}$  and  $C_4 = \{5, 7\}$  and intersection vertices are  $k_2 = 3$ ,  $k_3 = 5$  and  $k_4 = 5$ .

After having fixed the order of the cliques, we may assume that the vertices of  $G$  are labeled as follows: if  $|C_1 \cup \dots \cup C_i| = s_i$ , then  $C_1 \cup \dots \cup C_i = \{1, 2, \dots, s_i\}$ . In other words,  $C_1 = \{1, \dots, s_1\}$  and  $C_i \setminus \{k_i\} = \{s_{i-1} + 1, \dots, s_i\}$  for  $i > 1$ . The vertices on the graph in Figure 1 are labeled in this way. Now we let  $\Gamma \subset G$  be the spanning tree of  $G$  whose edges are  $\{j, k_2\}$  with  $j \in C_1$  and  $j \neq k_2$ , and for  $i = 1, \dots, r$  the edges  $\{j, k_i\}$  with  $j \in C_i$  and  $j \neq k_i$ . In our example the edges of  $\Gamma$  are  $\{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{5, 6\}$  and  $\{5, 7\}$ .

Let  $m + 1 = s_r$ . Then  $m + 1$  is the number of vertices of  $G$ . We now assign to  $\Gamma$  the following  $m \times m + 1$ -matrix  $A$  whose rows  $r_e$  correspond to the edges  $e$  of  $\Gamma$  as follows: we set  $r_e = -x_{1j}\varepsilon_j + x_{1k_2}\varepsilon_{k_2}$  for  $e = \{j, k_2\}$  and  $j \in C_1$  with  $j \neq k_2$ , and we set

$r_e = -x_{ij}\varepsilon_j + x_{ik_i}\varepsilon_{k_i}$  for  $e = \{j, k_i\}$  and  $j \in C_i$  with  $j \neq k_i$  and  $i > 1$ . Here  $\varepsilon_i$  denotes the  $i$ th canonical unit vector in  $\mathbb{R}^{m+1}$ .

The rows  $r_e$  can be naturally ordered according to the size of  $j$  in the edge  $e = \{j, k_i\}$ . Thus in our example we obtain the matrix

$$\begin{pmatrix} -x_{11} & 0 & x_{13} & 0 & 0 & 0 & 0 \\ 0 & -x_{12} & x_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & -x_{24} & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & -x_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{35} & -x_{36} & 0 \\ 0 & 0 & 0 & 0 & x_{45} & 0 & -x_{47} \end{pmatrix}$$

Our next goal is to show that our matrix  $A$  is a Hilbert–Burch matrix. We apply Theorem 1.5. For each edge  $\{i, j\} \in \Gamma$  the monomials  $u_{ij}$  and  $u_{ji}$  are, according to the choice of  $A$ , the following:

$$u_{jk_2} = -x_{1j}, \quad u_{k_2j} = x_{1k_2} \quad \text{for } j < k_2,$$

and for  $i = 2, \dots, r$

$$u_{k_i j} = x_{ik_i}, \quad u_{jk_i} = -x_{ij} \quad \text{for } k_i < j, \quad j \in C_i.$$

According to Theorem 1.5(b) we have to show that  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$  for all  $i < j$ . Assume first that  $i, j \notin \{k_2, \dots, k_r\}$ . Then  $u_{ib(i,j)} = -x_{ti}$  for  $i \in C_t$  and  $u_{je(i,j)} = -x_{sj}$  for  $j \in C_s$ . Thus in this case  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$ . In the second case let  $i \notin \{k_2, \dots, k_r\}$  and  $j \in \{k_2, \dots, k_r\}$ , let say  $j = k_s$ . Then  $b(i, j) = k_t$  for  $i \in C_t$  and so  $u_{ib(i,j)} = -x_{ti}$ . Suppose  $\{i, j\}$  is not an edge then  $e(i, j) = b(j, i) = b(k_s, i)$  is either  $k_{s+1}$  or  $k_{s-1}$ . Then  $u_{je(i,j)}$  is either  $-x_{(s)j}$  or  $-x_{(s-1)j}$ . On the other hand, if  $\{i, j\}$  is an edge, then  $e(i, j) = i$ , and so  $u_{je(i,j)} = u_{ji} = u_{k_s i} = x_{sj}$ . Thus in this case, too,  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$ . Finally assume that  $i, j \in \{k_2, \dots, k_r\}$ , and let  $i = k_{s_1}, k_{s_1+1}, \dots, k_{s_2} = j$  be the path from  $i$  to  $j$ . Then  $b(i, j) = k_{s_1+1}$  and  $e(i, j) = k_{s_2-1}$  so  $u_{ib(i,j)} = x_{s_1 i}$  and  $u_{je(i,j)} = -x_{(s_2-1)j}$ . Thus again in this case we have  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$ . Thus in all cases  $\gcd(u_{ib(i,j)}, u_{je(i,j)}) = 1$  for all  $i < j$ , as desired.

Let  $I$  be the codimension 2 ideal whose relation matrix is  $A$ , and let  $\{i, j\}$  be any edge of  $G$ . It remains to be shown that there exists a relation tree  $\Gamma'$  with  $\{i, j\} \in E(\Gamma')$ . If  $\{i, j\} \in E(\Gamma)$ , we are done. Now assume that  $\{i, j\} \notin E(\Gamma)$ . We may assume that  $\{i, j\} \in C_t$ . Let  $s = t$  if  $t > 1$ , and  $s = 2$  if  $t = 1$ . We replace the row  $-x_{tj}\varepsilon_j + x_{tk_s}\varepsilon_{k_s}$  of  $A$  by the difference of the rows

$$-x_{ti}\varepsilon_i + x_{tj}\varepsilon_j = (-x_{ti}\varepsilon_i + x_{tk_s}\varepsilon_{k_s}) - (-x_{tj}\varepsilon_j + x_{tk_s}\varepsilon_{k_s}),$$

and leave all the other rows of  $A$  unchanged. The new matrix  $A'$  is again a relation matrix of  $I$  and the tree  $\Gamma'$  corresponding to  $A'$  is obtained from  $\Gamma$  by removing the edge  $\{j, k_s\}$  and adding the edge  $\{i, j\}$ . This completes the proof of the theorem.  $\square$

## REFERENCES

- [1] W. Bruns, J. Herzog. On multigraded resolutions, Math. Proc. Cambridge Phil. Soc. **118**, 234-251 (1995).
- [2] W. Bruns, J. Herzog. *Cohen Macaulay rings*, Revised Edition, Cambridge, 1996.
- [3] G. A. Dirac. On rigid circuit graphs. Abh. Math. Sem. Univ. Hamburg, **38**, 71 - 76 (1961).

- [4] D. Eisenbud. *Commutative algebra; with a view towards algebraic geometry*, Graduate Texts Math., Springer. 1995.
- [5] J. Herzog, T. Hibi and X. Zheng. Dirac's theorem on chordal graphs and Alexander duality. *European J. Comb.* **25**(7), 949–960 (2004).
- [6] R. H. Villareal. *Monomial algebras*, Dekker, New York, 2001

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